FOCUS on Learning Problems in Mathematics is an inter-disciplinary journal, published jointly by the **Research Council on Mathematics Learning (RCML)** and the **Center for Teaching/Learning of Mathematics (CT/LM)**. The objective of FOCUS is to disseminate current research efforts designed to understand and/or influence factors that affect mathematics learning. Contributions from the fields of education, psychology, mathematics and medicine having the potential for impact on classroom or clinical practices are valued. Specifically, the types of manuscripts sought for **FOCUS** include reports of research on processes, techniques, tools and procedures useful for addressing problems in mathematics teaching and learning, descriptions of methodologies for conducting, and reporting and interpreting the results of various types of research, research-based discussions of promising techniques or novel programs, and scholarly works such as literature reviews, philosophical statements or critiques.

Manuscripts for publication should follow the guidelines in the **Publications Manual of American Psychological Association (Fifth Edition)**, and include a statement that the paper is not in submission to another journal. Five copies should be submitted to the editor: Dr. Jean Schmittau, School of Education and Human Development, State University of New York, P.O. Box 6000, Binghamton, NY 13902-6000. Manuscripts determined to be appropriate will be sent, with author identification removed, to at least two appropriate reviewers. Manuscripts will be reviewed for their uniqueness, significance, accuracy, and potential for impact.

Subscription: There are two ways to subscribe to FOCUS on Learning Problems in Mathematics. (1) A membership in RCML ($28.00 per year available at http://www.unlv.edu/RCML) includes all benefits of membership and a subscription to **FOCUS**. (2) A subscription through CT/LM: worldwide rates (in US funds) are $30.00, $45.00, or $55.00 for one, two, or three years for individuals; and $45.00, $60.00, or $70.00 for one, two, or three years for institutions. For back issues send $9.00 per copy to CT/LM. Outside North America add $10.00 for shipping and handling.

FOCUS is published four times a year: **Winter, Spring, Summer, Fall**. For advertising and publication information write to: **CENTER FOR TEACHING/LEARNING OF MATHEMATICS***, P.O. Box 3149, Framingham, Massachusetts 01701. (508) 877-7895 or (617) 235-7200.

*A list of other publications is available on request. CT/LM also publishes a monthly newsletter: **Math Notebook**. Yearly subscription to **Math Notebook** is $18.00 (ten issues), outside North America - $5.00 extra. A 10% discount is available for more than 10 copies delivered to the same address.

FOCUS

ON LEARNING PROBLEMS IN MATHEMATICS

Fall Edition 2004 ***Volume 26, No. 4***

Supported by Binghamton University Center for Science, Math, and Technology

TABLE OF CONTENTS

Learning Disabilities in Mathematics: A Review of the Issues and Children's Performance across Mathematical Tests 1
Seleshi Zeleke, University of Oslo, Norway

Middle School Mathematics Classroom Practices and Achievement: A TIMSS-R Analysis 19
James A. Telese, University of Texas, Brownsville

On Two Types of Learning (in Mathematics) and Implications for Teaching 31
Terrance Quinn, Ohio University Southern

Are Visual Perceptual Skills Related to Mathematics Ability in Second Through Sixth Grade Children? 44
Marjean Taylor Kulp, Michael J. Earley, G. Lynn Mitchell, Lisa M. Timmerman, Cara S. Frasco, and Meghan E. Geiger
The Ohio State University College of Optometry

Focus on Learning Problems in Mathematics
Fall Edition 2004, Volume 26; Number 4

Learning Disabilities in Mathematics: A Review of the Issues and Children's Performance Across Mathematical Tests

Seleshi Zeleke
University of Oslo, Norway

Abstract

In recent years, the research literature on mathematics disabilities (MD) has shown relative growth. The characteristics of children with MD have thus been investigated from different perspectives. The purpose of this paper was to review one aspect of this growing literature: the performance of children with MD on arithmetic facts and word problems. Alongside this primary concern, the paper examined some of the issues surrounding MD. The results suggest that because of the unresolved issue on definition, investigators have used different operational definitions for MD. Despite this variation, studies were consistent in showing that the MD/RD children's performances on both number facts and word problems were significantly worse than the performances of NA children. Results were partly inconsistent when it comes to differences between MD-only and NA children, however. Whereas most studies documented better performance for the NA children, some showed that the two groups had comparable performances on both number facts and word problems particularly when these tasks were not complex or timed. The implications for further research are discussed.

Introduction

For some years now, mathematics disabilities (MD) have been recognized as a type of learning disabilities (LD), as evidenced by the inclusion of mathematics in LD definitions (Bryant, Bryant, & Hammill, 2000). There is also general consensus among professionals in the field that MD is widespread in young children and that it has serious educational consequences (e.g., Bryant et al., 2000; Ginsburg, 1997; Jordan & Hanich, 2000; Jordan &

Steffe, L., & Gale, J. (Eds). (1995). *Constructivism in education*. Hillsdale, NJ: Erlbaum.
Tripp, J.S. (1998). Getting students to do homework. *Mathematics Teacher, 91*, 478-479.
Westat. (2000). *WesVar*. Rockville, MD: (Author).
Yamamoto, K., & Kulick, E. (2000). Scaling methodology and procedures for the TIMSS mathematics and science scales. In M.O. Martin, K.D. Gregory, & S.E. Stemler (Eds.), *TIMSS 1999 Technical Report* (pp. 235-263). Chestnut Hill, MA: International Study Center, Boston College.

Focus on Learning Problems in Mathematics
Fall Edition 2004, Volume 26; Number 4

On Two Types of Learning (in Mathematics) and Implications for Teaching

Terrance Quinn
Ohio University Southern

Abstract

A preliminary identification is made of two types of understanding – *nominal* and *explanatory.* The distinction has a bearing on teaching mathematics. To help illustrate the ideas, a detailed outline for an introduction to the dot product in linear algebra is given. Some contemporary issues in teaching are then discussed, particularly in regard to certain trends in contemporary textbook publication.

Introduction

I have been studying mathematics for most of my life so far. For me, one of the marked features of learning mathematics has been joy. In that I am not alone. In some fashion or another, colleagues frequently express similar sentiments. Of those who do not do so explicitly, there can be the testimony of their lively presence when they talk about mathematical discoveries.

Basically then, mathematics can be a happy science. Of course, this does mean that mathematics is easy, or that there is any lack of struggle required. Even the most gifted reach greatness only after prolonged effort. But, whether one is a beginner or a professional, growing in mathematics is essentially a happy occupation. In other words, from elementary puzzles to advanced theorems, there is the joy of discovery.

Motivation for writing this paper, however, partly comes from continuing reports to the contrary, from what is evidently an accumulating majority of students of all ages. It is troubling how regular and even acceptable it has become to hear expressions like "Oh no, not mathematics!", "I was never any good at mathematics.", "I hate mathematics.", or "Mathematics is boring." For some individuals, these feelings even can become a more or less permanent fear. Along with this widespread distaste for mathematics, there is the increasing lack of competence in basic mathematics skills.

Consider the familiar situation from College Algebra. A few weeks after an exam, of those students who happen to remember the formula for combining fractions, how many are able to explain the sum 1/3 + 1/5? A particular instance is the following: Recently, a very sincere first year student was telling me that, in his high school mathematics classes, he had had no problem with algebra, but he could not do problems involving particular ratios. We spent some time at the board chatting through some examples. And sure enough, for certain cases at least, he could factor algebraic expressions and "cancel" like-terms in rational expressions. But, where did the terms go once they were "canceled?" When trying to solve "real-world" problems, it became apparent that he did not have a basic grasp of numerical fractions. In particular, he was not aware of the helpful elementary school diagram that partitions a square both vertically and horizontally, and so reveals the rule for multiplication of fractions. I mention this student not because his problem is unusual, but because this type of problem is too usual. At the same time, its very prevalence does not imply a lacking in native ability of students. Instead, this is evidence that, despite positive discoveries made by scholars in mathematics education, certain flaws continue to influence teaching methods brought to the classroom.

Recently, some of these difficulties were the topic of a conference organized by the MAA (Conference to Improve College Algebra, Task Force on the First College-Level Mathematics Course, U.S. Military Academy, February 7-10, 2002; supported by the Historically Black Colleges and Universities (HBCU) Consortium for College Algebra Reform). Results of this conference were discussed in the May/June 2002 issue of the MAA publication *Focus,* in the article "An Urgent Call to Improve Traditional College Algebra Programs" (Small, 2002).

The first part of the present paper addresses a distinction that relates to the broader significance of the MAA recommendations. The distinction is between two types of insight, and would seem to pertain to all contexts of mathematical understanding. For, in all mathematical settings, there is the possibility of nominal understanding (where one comes to know how to use mathematical names) and of explanatory understanding (where one comes to know what mathematical names mean).

While the distinction between naming and explaining is not identical with, it is compatible with what has traditionally been described as the difference between "learning by rote" and "learning by understanding." So, in 1900, Felix Klein was struggling with mathematics education problems of his own day. Indeed, he was "at the forefront of a movement to reform mathematics education from rote learning to more meaningful mathematical learning" (McComas, 2000).

Following on the general distinction between nominal and explanatory understanding, there is the practical question of what to do. As a teacher, what teaching approach is one to take that exploits the natural occurrence of

these two types of understanding, and that is in harmony with the proven general advisory on development that emerged from the MAA conference? The second section of this paper, then, is an attempt to address this question somewhat by giving an outline for an approach to teaching the dot product. It is an approach that seems to have some of the desired features, and is one that my students have been enjoying.

There are several reasons for having selected the dot product for illustration. The formula $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta)$ links the algebra of vector coordinates to the Euclidean geometry of lengths and angles, and consequently is a key formula in basic mathematics that enters into numerous mathematical disciplines, both pure and applied. At the same time, it seems to be a formula that many students do not understand. Indeed, from conversations with professional engineers in both the USA and Canada, a common problem is that engineering graduates frequently have little grasp of the basic result. At the same time, it is a formula that is needed for many engineering processes that involve two or more parameters. And in my own experience in mathematics departments, it has been the rare mathematics student (undergraduate or graduate) who understands the formula. I have found, however, that basic insight leading to the formula is usually accessible to first year undergraduate students. Also, the result can serve as an entry to coordinate geometry.

One may begin with basic puzzles about right angle triangles and use strategic (ancient) diagrams that help reveal the Pythagorean formula. From there, how to extend the Pythagorean formula to other triangles can be brought out as a natural question. With clues and in-class preparation, I have found this to be a generally accessible homework problem for students, leading them to their own discovery of the Law of Cosines. Through examples, one can then help students realize that coordinates can be a convenient addition to the context. The dot-product can then emerge in a natural way that students tend to subsequently formalize with both ease and comfort. The relationship between the Law of Cosines and the dot product is, of course, a known mathematical result. Again though, the present purpose is not to offer a new mathematical result, but an approach to teaching the result that avoids rote learning and fosters the emergence of secure explanatory ("mathematically meaningful") understanding.

Note that I have included various exercises and questions in the written lesson - sometimes within the text, and some more formally as exercises. For convenience, answers immediately follow the stated questions. In the actual classroom situation, it usually takes me about three hours of classroom time to slowly tease out the key insights of the entire lesson, partly through classroom discussion and partly through home time for the students to work on the exercises.

In the third and last next section of this paper, remarks are given on some teaching issues, particularly in regard to certain trends in contemporary textbook publication. The discussion then leads back to the question of why so many students have such unpleasant experiences in basic mathematics classes. Suggestions are given for how one might make positive use of existing textbooks - including ones that are pedagogically problematic. The paper concludes with some general points regarding the large scale enterprise of mathematics education.

Two Types of Understanding

Consider the equation $y = 2x$. One could describe the curves of the printed letters, the cross of the x, and the lines for the "equals sign." Or, one might be able to divide and obtain, say, $y/2 = x$. But neither description of the patterned symbols, nor mere symbolic technique would, in themselves, get one any closer to the basic meaning of the otherwise elementary equation.

What is that meaning? The symbols (4, 8) and (153, 306) are not similarly described. In both cases, however, one may grasp the same relationship. One way to express this relationship is to call the first quantity an x and the second quantity a y; then y is proportional to x - in this case, by a factor of 2. Therefore, although basic, grasping proportionality is, evidently, a subtle moment in understanding. For in that insight one grasps a relationship between quantities; and that understanding goes beyond how things might be described in any particular case. Note, further, that once one has grasped the meaning of ratio and proportion, the insight readily can be repeated for a range of problems pertaining, for example, to classical Euclidean geometry, coordinate equations of lines, or even production ratios in an economy. Evidently, understanding basic connections is at once both precise and versatile.

This leads, therefore, to the following provisional definitions:

Definition: Nominal understanding grasps sensible and/or imaginable patterns (or compounds and associations of sensible and/or imaginable patterns) merely as sensible and/or imaginable.

Definition: Explanatory understanding grasps terms, and connections between terms.

In nominal understanding, therefore, our understanding would be of how things relate to us; while in explanatory understanding, our understanding would be of how terms relate to terms. Furthermore, with these definitions, it would seem that in order for understanding to be mathematically meaningful as such, it would necessarily be explanatory.

As mentioned briefly in the Introduction, it is not being suggested that nominal understanding does not have its vital and even permanent function in mathematics. Indeed, it would seem that nominal understanding has at least two normative roles: (1) At any stage of development, one can explain for oneself only some limited cross-section of mathematical results; hence techniques will always be needed. (2) With regard to the possibility of further development, understanding the use of names, symbols or diagrams can help provide one with the patterned data needed in order to reach explanatory insight.

Squares and Products

This section of the paper outlines a possible lesson that is intended to foster mathematical (explanatory) understanding of the dot product. We begin with images and the generation of questions.

Imagine, then, that you are at one comer of a rectangular field. Perhaps you've been working through a summer's morning and there is a jug of water at the far comer of the field, diagonally opposite (Figure 1).

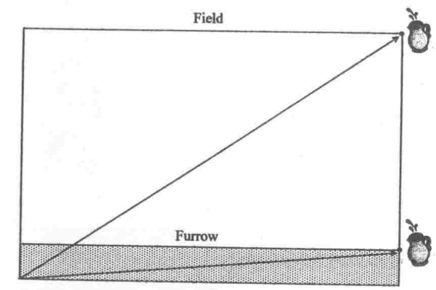

Figure 1. How does the distance across the field relate to the lengths of the sides of the field?

What would be the shortest path for reaching the water? Commonly enough, the path of choice would be along one's line of sight. In other words, cut across the diagonal of the field and head straight for the water. So there can be the implicit insight that following the sides of the field would be a greater total distance than simply walking in a straight line.

Suppose that there is another jug of water, at the corner of a narrow furrow that runs the length of the same field, along one side (Figure 1). Apparently, it would matter less in this case, whether one walks diagonally across the long furrow straight for the water, or simply walks along the one long side of the furrow and then takes a few extra steps to get to the water.

It would seem, therefore, that the length of the diagonal depends on the lengths of the sides. So imagine a rectangle whose sides are of length a and b, and whose diagonal is of length c. A question that can arise is how these lengths are related to each other. In symbols, we might write: c ? a, b .

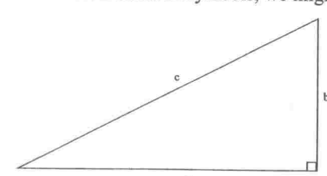

Figure 2. How does the hypotenuse c relate to the perpendicular sides a and b?

Solutions to special cases of this kind of puzzle were known in ancient India, China and other cultures. In India, c. 500 B.C, the sides are equal, with $a = 1$ and $b = 1$ (Katz, 1998).

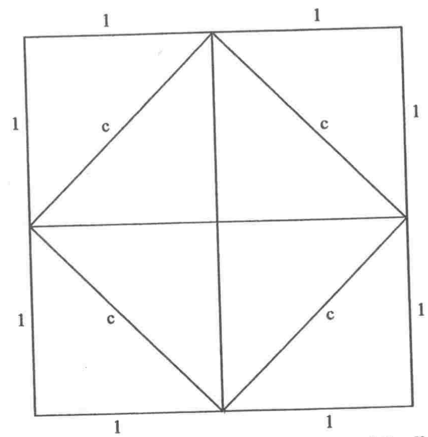

Figure 3. Diagram from India, c. 500 B.C.: The square of the diagonal is 2.

Take a few moments and think about what the diagram holds. Identify which parts of the diagram represent the unknown *c*. Curiously, while we are looking for a length *c*, the diagram involves areas. That is a clue. How do the areas fit together? Another clue: The unknown *c* is the length of a side of a square that is situated inside a larger square.

Look at how the areas may be subdivided into congruent triangles. From the diagram, the area of the larger square is four square units, while the area of the smaller square is four triangles that add to two square units. So $c^2 = 1^2 + 1^2 = 2$; and *c* is the square root of the number 2.

Another special case that was known to the ancients is when $a = 3$ and $b = 4$. The diagram (Figure 4) is from China, c. 200 B.C. (Katz, 1998, p.34). Perhaps in this case you already know what *c* is? The main question now, however, is not merely to know the answer, but to (mathematically) understand the answer. In other words, what are the terms and relations?

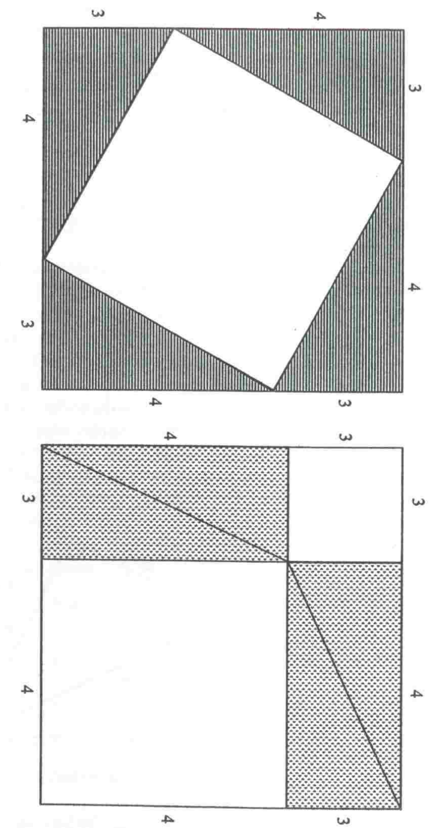

Figure 4. Diagram from China, c. 200 B.C.: The square of the diagonal is the square of 3 plus the square of 4.

Again, notice that the unknown length c is represented by the side of a square that is interior to a larger square. The larger square has dimensions (3 + 4) x (3 + 4). One way to key into the problem is to subdivide the large square in two different ways, and then compare.

In the first version, the (3 + 4) x (3 + 4) square area consists of four triangles together with the interior square whose area is c^2. In the second version, the total area consists of four triangles and two squares. How do the triangles of the second version compare with the triangles of the first version?

The triangles are congruent. Hence, removing the triangles from each version, the remaining area in the first must equal the remaining area in the second. In other words, $c^2 = 3^2 + 4^2 = 25$; and so the positive length $c = 5$.

Exercise: Suppose that a right triangle has base a, side b and diagonal c. Construct a square of dimensions $(a + b)$ x $(a + b)$. Subdivide the square in two ways. Hence, obtain a general solution to the question $c?a, b$. (Of course, this is the Pythagorean Formula, $c^2 = a^2 + b^2$.)

Now, as some students will point out (or as may easily be illustrated in diagrams) not all triangles are right triangles. So again, there can be the question $c?a,b$, but now for an arbitrary triangle. A clue can be obtained by looking to the special case where $a = b = 1$. As above, start with a and b as the lengths of the perpendicular sides of a right triangle. Rotate the sides for a and b to an oblique angle (larger than perpendicular). The resulting new hypotenuse evidently is larger than the hypotenuse for the right angle. On the other hand, if the angle between a and b is rotated into an acute angle (less than perpendicular), then the resulting diagonal evidently is less than for the right triangle. Putting these two observations together, a clue is the possible dependence of the solution on the angle between the sides for a and b.

Following up on this clue, let's represent the angle by the Greek letter theta θ (Figure 5).

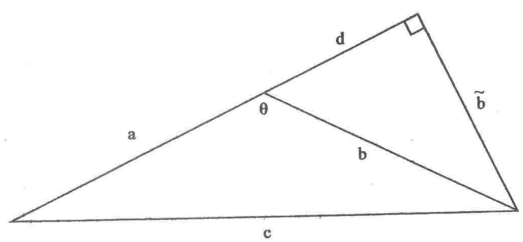

Figure 5. Extending the Pythagorean formula to arbitrary triangles.

From the example, we might expect the general case to be something like $c^2 = a^2 + b^2 + x$, where x is some correction term depending on the angle θ. One approach might be to try to relate the general case to a construction involving what we already know about right triangles (Figure 5), in a way that would somehow reveal the correction term.

In Figure 5, there are two right triangles. Do you see them? For the large right triangle, we have $c^2 = (a + d)^2 + \tilde{b}^2$ (d for difference). But d, b and $\tilde{b}$ also form a right triangle. So, $b^2 = \tilde{b}^2 + d^2$. Now, remember that we are looking for a relationship involving a, b and c. Can you find a way to get closer to a formula involving only a, b and c ? Notice that $\tilde{b}^2 = b^2 - d^2$; and consider the formula $c^2 = (a + d)^2 + \tilde{b}^2$.

By substituting for $\tilde{b}^2$, we get $c^2 = (a + d)^2 + b^2 - d^2$. Notice, however, that

the d so far remains unaccounted for. Let's keep that in mind, but first carry on with what we have - to see how far that will take us. Expanding the last expression gives $c^2 = a^2 + b^2 + 2ad$. So, we find that there is indeed a correction term, namely, $x = 2ad$. It also turns out that the d^2 terms are no longer present. So we need only determine d itself. It is left as an exercise for the student to use the given diagram to identify d in terms of the angle θ and so get that $2ad = -2ab \cos(\theta)$. (Notice the minus sign!) Our final result, therefore, is that $c^2 = a^2 + b^2 - 2ab \cos(\theta)$. This generalization of the Pythagorean Formula is called The Law of Cosines.

Exercise: Draw the appropriate diagrams and think about the following cases: (i) $\theta = 90°$; (ii) $0 < \theta < 90°$; (iii) $\theta = 0°$; (iv) $90° < \theta < 180°$; and (v) $\theta = 180°$. Notice how the minus sign and the angle together affect the quantity $a^2 + b^2 - 2ab\cos(\theta)$.

Let's now bring this result into the more modern setting of coordinate geometry (Fermat and Descartes, 17th century) and linear algebra (Cayley, 19th century) (Burton, 1999). In coordinate geometry, points in a plane are no longer merely points, but are conveniently located by two distances from an origin - one distance for each of two axes. So three points on a plane are determined by three pairs of coordinates (x_0, y_0), (x_1, y_1) and (x_2, y_2).

One of the objectives in coordinate geometry is to express geometric relations in terms of the coordinates of points. In particular, we may ask how The Law of Cosines can be expressed for the triangle located in the plane by the three points $P_0 = (0,0)$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ (Figure 6).

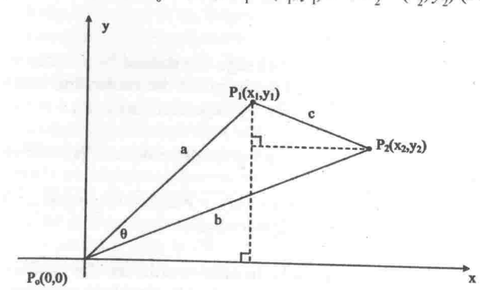

Figure 6. Coordinates of vertices of an arbitrary triangle, with one vertex at the origin.

Exercise: Let the distances be $P_0P_1 = a$, $P_1P_2 = c$ and $P_0P_2 = b$. Notice how P_0 and P_1 determine the vertices of a right triangle with sides x_1, y_1 and diagonal a. Hence, we may use the Pythagorean Formula to get that $a^2 = x_1^2 + y_1^2$. Similarly, $b^2 = x_2^2 + y_2^2$; and it is left as an exercise for the student to obtain $c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Writing the terms of the Law of Cosines using the coordinate representations just obtained, and canceling like terms, we get that $x_1x_2 + y_1y_2 = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \cos(\theta)$. Recall that the objective is to use algebra on the coordinates. This last expression certainly is the right sort of result. But, perhaps more can be said? Look at this with an "eye for algebra." In other words, do you see any pattern in the algebraic operations?

The left side of the last equation, as well as each of the terms inside the radicals, are obtained by multiplying x's and y's respectively, and then adding. From that observation, we can define the following "product": For any two pairs of numbers (x, y) and $(\bar{x}, \bar{y})$, $(x, y)\bullet(\bar{x}, \bar{y}) = x\bar{x} + y\bar{y}$. This is called the dot product, or scalar product. For present purposes, one may think of two pairs of numbers being combined to "produce" something new. It is called "dot" product for the symbol used. It is called "scalar" product, because what is produced is a number, and in some cases, "scalar" is another word for number.

Exercise: Express $x_1x_2 + y_1y_2 = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \cos(\theta)$ in terms of this dot product. Hence obtain $(x_1, y_1)\bullet(x_2, y_2) = \sqrt{(x_1, y_1)\bullet(x_1, y_1)}\sqrt{(x_2, y_2)\bullet(x_2, y_2)} \cos(\theta)$.

In linear algebra, an ordered pair of numbers typically represents a vector, which in the present setting of coordinates can be taken to mean a change in location. For example, on a geographical map scaled in miles, the vector $\mathbf{v}$ = (3,4) would not represent a location, but would represent a change in location, 3 miles east and 4 miles north. This one vector, then, could represent that change, starting from any location, whether it be 3 miles east and 4 miles north of Boston City Hall, 3 miles east and 4 miles north of Atlanta City Hall, or wherever.

Exercise: What is the distance of change determined by a vector $\mathbf{v} = (x, y)$? The distance of change is called the length of the vector, and is denoted $\|\mathbf{v}\|$. ($\|\mathbf{v}\| = \sqrt{x^2 + y^2}$. (Hint: Use the Pythagorean Formula.)

Exercise: Express $\|\mathbf{v}\| = \sqrt{x^2 + y^2}$ using the dot product. ($\|\mathbf{v}\| = \sqrt{\mathbf{v}\bullet\mathbf{v}}$)

Exercise: Express $(x_1, y_1)\bullet(x_2, y_2) = \sqrt{(x_1, y_1)\bullet(x_1, y_1)}\sqrt{(x_2, y_2)\bullet(x_2, y_2)} \cos(\theta)$ using vector notation. ($\mathbf{v}\bullet\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos(\theta)$)

Recall the origin of this formula. In other words, the dot product formula essentially is the vector formulation of the classical Law of Cosines.

Further Remarks and Implications for Teaching

I have at hand three different standard College Algebra books from major publishers, dated 2001, 2002 and 2003, respectively. Looking to the sections on parabolas, they each begin with some equivalent of: "This is the equation of a parabola." In each text the parabola sections consist of various examples and exercises on how to plot points on graphs. Note that in all three texts the

identical approach is taken for the exponential functions as well.

Now, it is not my purpose to review these texts. Certainly there are many fine features to each. What these three books illustrate, though, is an on-going trend in the mathematics textbook industry. In the context of the present article, it is possible now to more precisely identify certain mathematical difficulties of that trend.

In the first place, the approach taken by these three textbooks for parabolas does not promote "meaningful mathematical learning" (McComas, 2000). This may seem harsh. But, using the name "parabola," plotting points and being familiar with certain diagrams and shapes are in themselves mere techniques. These things do require nominal understanding (as defined in the first section of this paper). But merely learning these techniques does not require appreciation of any mathematical connection. Part of the challenge for the mathematics teacher, therefore, comes from the fact that the definition of the parabola is an answer to a question. But, what is a question (or directed sequence of questions) that can lead to that answer?)While appealing to history can be helpful, this does not mean that one would in all cases need to try to reproduce the identical questions from antiquity. There can be many inroads to the same insight. The point is that (whether for the dot product, the parabola, fractions, etc.) finding some directed set of questions and insights that converges on the discovery of the key mathematical relationship is essential. Indeed, without such questions and insights, a stated definition can be only a collection of words.

Coupled with the problematic trend in textbook writing that emphasizes mere symbolic technique is an emphasis of some of the main publishing houses to have technology based instruction as a necessary part of the standard elementary mathematics textbook. The MAA does advocate the "appropriate use of technology." And certainly there are well known uses of technology in learning and teaching mathematics. For example, where calculations by hand might otherwise be somewhat impractical, students often discover the meaning of convergence of a sequence of finite series, if they use calculators to more easily obtain several partial sums. Indeed, when enough terms are obtained, different convergence rates can be revealed. In undergraduate Differential Equations courses, once slope fields are understood, a graphing calculator (or program such as ODE Architect, etc.) can be used to provide examples that help illustrate varieties of trajectory patterns. In particular, this can help a student grow in their capacity to creatively imagine ranges of possibilities. Or, in Linear Algebra, once the basic concepts are well grasped by the student, computer programs can be used to illustrate the ideas with systems of equations involving large numbers of equations and variables. However, whether it is the technology of pen-and-paper or of keyboard-and -PC, in order for there to be mathematical understanding, it is necessary that there be connections between terms. So, not to exclude other possible uses of technology, "appropriate use of technology" for mathematics as such would be specifically directed to the generation of whatever data (symbolic, diagram, etc.) might be helpful for the emergence of explanatory mathematical understanding.

In this light, it is hard to see that technique oriented computer technology is essential to the occurrence of the basic insights proper to elementary mathematics such as College Algebra and Linear Algebra. Again, this is not to deny the possible usefulness of computer based technology. As history shows, however, independently of computer technology, basic understanding of key connections frequently can be obtained with judiciously selected puzzles and appropriate dynamic images.

It is also now possible to make some comment on the often unpleasant experiences of mathematics students, mentioned in the Introduction. For, in view of the distinction between explanatory and nominal understanding, it follows that "mathematics" through a technique oriented textbook approach is experience with "mathematics" in name only. Nor does technique on its own tend to be very interesting. What I hear regularly from frustrated students is that they did calculations a certain way because "That is what we were told to do." So, besides lacking mathematical content, the symbolic approach can effectively eliminate the occurrence of any wonder at all, whether toward nominal or explanatory understanding. Finally then, when basic mathematical insight is missing, rules for symbols can seem arbitrarily complex, and difficult to remember. Not surprisingly, students can be less than enchanted with the obscure, often frightening, and non mathematical enterprise.

There is then an immediate and practical question. For until such time as adequate reform becomes generally effective, one will need to make do with whatever books happen to be available. Fortunately, in numerous texts, many of the key ingredients already will be present in some way - though frequently hidden within axiomatics, in examples at the ends of sections, exercises at the ends of chapters, as appendices, etc. So, what is both needed and possible is a type of reading, where as a teacher one not only understands the results in their axiomatic context, but where one begins to seek out, "uncover," advert to and identify what we might call "developmental sequences of question and insight." It follows that while future textbook writing will need to be transformed in root and orientation, already we can begin to gradually effect a transformation of our classrooms.

Today's problems in mathematics education are historically conditioned within the mathematical community. Reaching toward a comprehensive solution, therefore, is a large-scale community project that involves reflection upon the complex dynamism of human enquiry. So, while much has been accomplished in the education fields, a verifiable general account of mathematical development is a goal for the future. Note that this is not grounds for consternation, but rather points to the profundity and worthwhileness of the objective. Consequently, we need not expect any quick solution. There will be no single new teaching technique to bring us past these difficulties. Indeed, such an approach would simply replace one set of techniques with another. Instead, eventually what might be possible are results that will, in their own way, partly consist of invariant connections and so, by the same token, be amenable to local needs and creativity. (As observed by

Small (2002), the problem always "is local in nature.") In this invitingly large context, I recall Klein's advice, which would seem to apply equally to understanding within mathematics as to the understanding of mathematics as a human enterprise: "slowly to higher things" (Klein, 1925, p. 268).

The present paper, therefore, is intended as a modest report based on a few selected examples. In my experience, however, I have been finding that adverting to such elementary examples has been increasingly helpful in furthering my own slow development as a teacher. Moreover, the approach of seeking data in this way may be more generally useful. I would think, for instance, that any viable theory of mathematical understanding should be verifiable in one's own moments of mathematical understanding. Perhaps, then, these brief notes will be of some use to others in the community teaching field.

References

Burton, D.M. (1999). *The History of Mathematics: An Introduction*. New York: WCB McGraw-Hill.

Katz, V.J. (1998). *A History of Mathematics: An Introduction,* (2nd ed.). Reading Massachusetts: Addison Wesley Longman, Inc.

Klein, F. (1925). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis.* (E.R. Hedrick and C.A. Noble, trans. from the third German edition). New York: Dover Publications.

McComas, K.K. (2000). Felix Klein and the NCTM's standards: A mathematician considers mathematics education. *Mathematics Teacher, 93*(8), 714-717.

Small, D. (2002, May/June). An urgent call to improve traditional college algebra programs. *Focus: The Newsletter of the Mathematical Association of America,* 12-13.

Author Note

Terrance J. Quinn, Department of Mathematics, Ohio University Southern.

I would like to thank Leslie Gannon for making the diagrams used in this paper.

Terrance J. Quinn, Department of Mathematics, Ohio University Southern, 1804 Liberty Avenue, Ironton, Ohio 45638, USA. E mail: quinnt@ohio.edu

understanding.

In this light, it is hard to see that technique oriented computer technology is essential to the occurrence of the basic insights proper to elementary mathematics such as College Algebra and Linear Algebra. Again, this is not to deny the possible usefulness of computer based technology. As history shows, however, independently of computer technology, basic understanding of key connections frequently can be obtained with judiciously selected puzzles and appropriate dynamic images.

It is also now possible to make some comment on the often unpleasant experiences of mathematics students, mentioned in the Introduction. For, in view of the distinction between explanatory and nominal understanding, it follows that "mathematics" through a technique oriented textbook approach is experience with "mathematics" in name only. Nor does technique on its own tend to be very interesting. What I hear regularly from frustrated students is that they did calculations a certain way because "That is what we were told to do." So, besides lacking mathematical content, the symbolic approach can effectively eliminate the occurrence of any wonder at all, whether toward nominal or explanatory understanding. Finally then, when basic mathematical insight is missing, rules for symbols can seem arbitrarily complex, and difficult to remember. Not surprisingly, students can be less than enchanted with the obscure, often frightening, and non mathematical enterprise.

There is then an immediate and practical question. For until such time as adequate reform becomes generally effective, one will need to make do with whatever books happen to be available. Fortunately, in numerous texts, many of the key ingredients already will be present in some way - though frequently hidden within axiomatics, in examples at the ends of sections, exercises at the ends of chapters, as appendices, etc. So, what is both needed and possible is a type of reading, where as a teacher one not only understands the results in their axiomatic context, but where one begins to seek out, "uncover," advert to and identify what we might call "developmental sequences of question and insight." It follows that while future textbook writing will need to be transformed in root and orientation, already we can begin to gradually effect a transformation of our classrooms.

Today's problems in mathematics education are historically conditioned within the mathematical community. Reaching toward a comprehensive solution, therefore, is a large-scale community project that involves reflection upon the complex dynamism of human enquiry. So, while much has been accomplished in the education fields, a verifiable general account of mathematical development is a goal for the future. Note that this is not grounds for consternation, but rather points to the profundity and worthwhileness of the objective. Consequently, we need not expect any quick solution. There will be no single new teaching technique to bring us past these difficulties. Indeed, such an approach would simply replace one set of techniques with another. Instead, eventually what might be possible are results that will, in their own way, partly consist of invariant connections and so, by the same token, be amenable to local needs and creativity. (As observed by

Small (2002), the problem always "is local in nature.") In this invitingly large context, I recall Klein's advice, which would seem to apply equally to understanding within mathematics as to the understanding of mathematics as a human enterprise: "slowly to higher things" (Klein, 1925, p. 268).

The present paper, therefore, is intended as a modest report based on a few selected examples. In my experience, however, I have been finding that adverting to such elementary examples has been increasingly helpful in furthering my own slow development as a teacher. Moreover, the approach of seeking data in this way may be more generally useful. I would think, for instance, that any viable theory of mathematical understanding should be verifiable in one's own moments of mathematical understanding. Perhaps, then, these brief notes will be of some use to others in the community teaching field.

References

Burton, D.M. (1999). *The History of Mathematics: An Introduction*. New York: WCB McGraw-Hill.

Katz, V.J. (1998). *A History of Mathematics: An Introduction,* (2nd ed.). Reading Massachusetts: Addison Wesley Longman, Inc.

Klein, F. (1925). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis.* (E.R. Hedrick and C.A. Noble, trans. from the third German edition). New York: Dover Publications.

McComas, K.K. (2000). Felix Klein and the NCTM's standards: A mathematician considers mathematics education. *Mathematics Teacher, 93*(8), 714-717.

Small, D. (2002, May/June). An urgent call to improve traditional college algebra programs. *Focus: The Newsletter of the Mathematical Association of America,* 12-13.

Author Note

Terrance J. Quinn, Department of Mathematics, Ohio University Southern.

I would like to thank Leslie Gannon for making the diagrams used in this paper.

Terrance J. Quinn, Department of Mathematics, Ohio University Southern, 1804 Liberty Avenue, Ironton, Ohio 45638, USA. E mail: quinnt@ohio.edu

Steffe, L., & Gale, J. (Eds). (1995). *Constructivism in education*. Hillsdale, NJ: Erlbaum.
Tripp, J.S. (1998). Getting students to do homework. *Mathematics Teacher, 91*, 478-479.
Westat. (2000). *WesVar*. Rockville, MD: (Author).
Yamamoto, K., & Kulick, E. (2000). Scaling methodology and procedures for the TIMSS mathematics and science scales. In M.O. Martin, K.D. Gregory, & S.E. Stemler (Eds.), *TIMSS 1999 Technical Report* (pp. 235-263). Chestnut Hill, MA: International Study Center, Boston College.

Focus on Learning Problems in Mathematics
Fall Edition 2004, Volume 26; Number 4

On Two Types of Learning (in Mathematics) and Implications for Teaching

Terrance Quinn
Ohio University Southern

Abstract

A preliminary identification is made of two types of understanding – *nominal* and *explanatory*. The distinction has a bearing on teaching mathematics. To help illustrate the ideas, a detailed outline for an introduction to the dot product in linear algebra is given. Some contemporary issues in teaching are then discussed, particularly in regard to certain trends in contemporary textbook publication.

Introduction

I have been studying mathematics for most of my life so far. For me, one of the marked features of learning mathematics has been joy. In that I am not alone. In some fashion or another, colleagues frequently express similar sentiments. Of those who do not do so explicitly, there can be the testimony of their lively presence when they talk about mathematical discoveries.

Basically then, mathematics can be a happy science. Of course, this does mean that mathematics is easy, or that there is any lack of struggle required. Even the most gifted reach greatness only after prolonged effort. But, whether one is a beginner or a professional, growing in mathematics is essentially a happy occupation. In other words, from elementary puzzles to advanced theorems, there is the joy of discovery.

Motivation for writing this paper, however, partly comes from continuing reports to the contrary, from what is evidently an accumulating majority of students of all ages. It is troubling how regular and even acceptable it has become to hear expressions like “Oh no, not mathematics!”, “I was never any good at mathematics.”, “I hate mathematics.”, or “Mathematics is boring.” For some individuals, these feelings even can become a more or